GEOMETRIC PROPERTIES OF THE JACOBIANS OF A CERTAIN SYSTEM OF FUNCTIONS.*

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1. In the proofs of the existence theorem for implicit functions of several variables the assumption is made that the corresponding Jacobian does not vanish for any point within the interval for which the functions are defined.† Also in the general theory of analysis situs as developed so far,‡ the cases in which the Jacobians vanish simultaneously with the corresponding functions are excluded.

It may be expected that the vanishing of a Jacobian and its functions for certain values of the variables signifies a particular property of these functions which deserves to be investigated.§

It is the purpose of this paper to show the importance of such cases by studying the geometric properties of a certain system of functions and their Jacobians.

2. Let

$$x = \phi(t), \qquad y = \psi(t) \tag{1}$$

be two real, distinct, uniform, continuous and singly periodic functions of the real parameter t and with the same period w. As is well known, in a Cartesian plane (1) represents a closed *Jordan curve.*‖

We assume furthermore that $\phi'(t)$ and $\psi'(t)$ are also continuous within the period-interval and do not vanish simultaneously for any value of t. In other words, the curve as represented by (1) is analytic throughout and its only singular points, if there are any, are multiple points.

3. If z_1, z_2, z_3, z_4 designate four independent variables, consider now the three functions

$$\begin{cases} F_1(z_1, z_2, z_3, z_4) = \phi(z_1) - \phi(z_2) + \phi(z_3) - \phi(z_4), \\ F_2(z_1, z_2, z_3, z_4) = \psi(z_1) - \psi(z_2) + \psi(z_3) - \psi(z_4), \\ F_3(z_1, z_2, z_3, z_4) = [\phi(z_1) - \phi(z_3)][\phi(z_2) - \phi(z_4)] \\ \qquad\qquad + [\psi(z_1) - \psi(z_3)][\psi(z_2) - \psi(z_4)]. \end{cases} \tag{2}$$

* Read before the American Mathematical Society in Chicago, Dec. 26, 1913.

† Genocchi-Peano, Differentialrechnung (1899), pp. 147–152; Bliss, "A new proof of the existence theorem of implicit functions," Bulletin of the Am. Math. Soc., vol. XVIII, pp. 175–179 (1912).

‡ Poincaré: "Analysis situs," Journal de l'École Polytéchnique, 2nd ser., vol. 1, pp. 1–121.

§ Clements, "Implicit functions defined by equations with vanishing Jacobian," Bulletin of the Am. Math. Soc., vol. XVIII, pp. 451–456 (1912).

‖ "Osgood, Lehrbuch der Funktionentheorie," vol. 1, 2nd ed., pp. 146–150 (1913).

They are evidently uniform, continuous, analytic and co-periodic for all values of the variables. To any four values $z_1 < z_2 < z_3 < z_4$ within a period-interval correspond on the curve (1) the vertices of an inscribed quadrangle $A_1A_2A_3A_4$ which follow each other in the same order as the entire curve is described in the same sense. If for any set of values of the z's $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, then the quadrangle will be a rhomb. As will be proved elsewhere, there are always an infinite number of rhombs inscribable in the given curve. For any given direction and the corresponding orthogonal direction there is at least one rhomb inscribable and with its diagonals parallel to the pair of orthogonal directions. There is always a continuous curve between the points of tangency of the two extreme tangents to the curve parallel to the given direction and which is the locus of mid-points of a continuous system of chords parallel to the same direction. I shall call such a curve a *median* M_τ of the given closed curve C (1), with respect to the direction τ. *If C_1C_2 is one of the chords of the system cutting M_τ in M, then the tangents to C at C_1 and C_2 and to M_τ at M are concurrent.*

With every pair τ, σ of orthogonal directions are associated two medians M_τ and M_σ which always intersect in at least one point. The points of intersection of M_τ and M_σ are evidently the centers of inscribed rhombs with diagonals parallel to τ and σ. *When M_τ and M_σ touch each other at some point R, then the tangents to C at A_1 and A_3 intersect in a point T_1 and those at A_2, A_4 in a point T_2 so that T_1, R, T_2 are collinear.*

4. In the system of simultaneous equations

$$F_1 = 0, \qquad F_2 = 0, \qquad F_3 = 0, \tag{3}$$

consider any one of the four variables, for instance z_4 as independent. According to the theorem on implicit functions,* in the neighborhood of any set z_1, z_2, z_3, z_4 satisfying (3) and for which the Jacobian

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \tag{4}$$

$$= \begin{vmatrix} \phi'(z_1) & -\phi'(z_2) & \phi'(z_3) \\ \psi'(z_1) & -\psi'(z_2) & \psi'(z_3) \\ \left\{\begin{matrix} \phi'(z_1)[\phi(z_2)-\phi(z_4)] \\ +\psi'(z_1)[\psi(z_2)-\psi(z_4)] \end{matrix}\right\}, & \left\{\begin{matrix} \phi'(z_2)[\phi(z_1)-\phi(z_3)] \\ +\psi'(z_2)[\psi(z_1)-\psi(z_3)] \end{matrix}\right\}, & \left\{\begin{matrix} -\phi'(z_3)[\phi(z_2)-\phi(z_4)] \\ -\psi'(z_3)[\psi(z_2-\psi(z_4)] \end{matrix}\right\} \end{vmatrix}$$

does not vanish, it is possible to represent z_1, z_2, z_3 as uniform, continuous and analytic functions of z_4.

* Bliss, loc. cit.

Equations (3) and (4) vanish simultaneously only for a finite number of sets z_1, z_2, z_3, z_4; hence for any domain of four space z_1, z_2, z_3, z_4 within which

$$F_1 = 0, \qquad F_2 = 0, \qquad F_3 = 0,$$

and

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \neq 0,$$

the z_i's ($i = 1, 2, 3, 4$) form a continuous set. Geometrically to such a domain corresponds a continuous set of inscribed rhombs.

5. I shall now investigate the case in which the Jacobian (4) vanishes simultaneously with (3). Without loss of generality it may be assumed that the axes of the rhomb in this case coincide with the coordinate axes, so that $A_1(z_1)$, $A_3(z_3)$ are on the x- and $A_2(z_2)$, $A_4(z_4)$ on the y-axis. Designating the coordinates of A_1, A_2, A_3, A_4 respectively by $(a, 0)$; $(0, b)$; $(-a, 0)$; $(0, -b)$, the condition that (4) vanishes becomes

$$\text{(5)} \qquad \begin{vmatrix} \phi'(z_1) & -\phi'(z_2) & \phi'(z_3) \\ \psi'(z_1) & -\psi'(z_2) & \psi'(z_3) \\ 2b\psi'(z_1) & 2a\phi'(z_2) & -2b\psi'(z_3) \end{vmatrix} = 0.$$

Assuming $\phi'(z_i) \neq 0$ ($i = 1, 2, 3$) which, by choosing the x- and y-axis properly, does not imply a special case; dividing the first, second and third column by $\phi'(z_1)$, $\phi'(z_2)$, $\phi'(z_3)$ respectively, and designating the slopes of the tangents to C at A_1, A_2, A_3 by m_1, m_2, m_3, (5) reduces to

$$\text{(6)} \qquad \phi'(z_1)\phi'(z_2)\phi'(z_3) \begin{vmatrix} 1 & -1 & 1 \\ m_1 & -m_2 & m_3 \\ bm_1 & a & -bm_3 \end{vmatrix} = 0,$$

or explicitly, if $\phi'(z_1) \neq 0$, $\phi'(z_2) \neq 0$, $\phi'(z_3) \neq 0$, to

$$\text{(7)} \qquad a(m_1 - m_3) + b(m_1m_2 - 2m_1m_3 + m_2m_3) = 0.$$

6. To find a geometric interpretation for this equation, consider the corresponding rhomb A_1, A_2, A_3, A_4 and apply first the dilatation

$$\text{(8)} \qquad \begin{aligned} x' &= (1 + e_1)x \\ y' &= (1 + e_2)(y + b) - b, \end{aligned}$$

and second the rotation

$$\text{(9)} \qquad \begin{aligned} x'' &= x' \cos\theta - (y' + b)\sin\theta \\ y'' &= x' \sin\theta + (y' + b)\cos\theta \end{aligned}$$

with A_4 as a center, Fig. 1. The lines $x = 0$ and $y = -b$ are invariant in the dilatation. After this dilatation and rotation the coordinates of the rhomb in the new position are for

$$A_1''\begin{cases}(1+e_1)a\cos\theta-(1+e_2)b\sin\theta,\\(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b,\end{cases}$$

$$A_2''\begin{cases}-(1+e_2)2b\sin\theta,\\(1+e_2)2b\cos\theta-b,\end{cases}$$

$$A_3''\begin{cases}-(1+e_1)a\cos\theta-(1+e_2)b\sin\theta,\\-(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b,\end{cases}$$

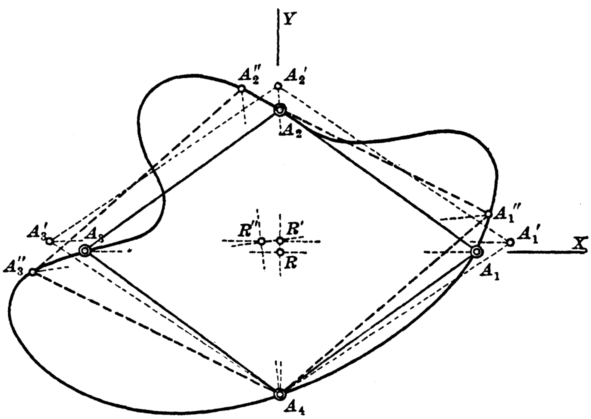

FIG. 1.

referred to the original xy-plane. The slopes μ_1, μ_2, μ_3 of the lines A_1A_1'', A_2A_2'', A_3A_3'' are

$$\mu_1=\frac{(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b}{(1+e_1)a\cos\theta-(1+e_2)b\sin\theta-a},$$

$$u_2=\frac{(1+e_2)b\cos\theta-b}{(1+e_2)b\sin\theta},$$

$$\mu_3=\frac{-(1+e_1)a\sin\theta+(1+e_2)b\cos\theta-b}{-(1+e_1)a\cos\theta-(1+e_2)b\sin\theta+a}.$$

Substituting these values in the reduced Jacobian expression (7)

$$J_4=a(\mu_1-\mu_3)+b(\mu_1\mu_2-2\mu_1\mu_3+\mu_2\mu_3), \tag{10}$$

after some reductions, we get

$$J_4=\frac{2a_1^2b_2(e_2-e_1)\left\{2+e_1+e_2-\dfrac{e_1e_2}{\cos\theta-1}\right\}}{(1+e_2)\left\{b^2(1+e_2)^2\dfrac{\sin^2\theta}{\cos\theta-1}+a_1^2\left[1-\cos\theta+2e_1\cos\theta+\dfrac{e_1^2}{1-\cos\theta}\cos^2\theta\right]\right\}}. \tag{11}$$

Considering e_1 and e_2 as infinitesimals of the first order and $e_2 = \alpha e_1$ (α any finite integer), the limits of $e_1 e_2/\cos\theta - 1$, $\sin^2\theta/\cos\theta - 1$, $e_1^2\cos^2\theta/\cos\theta - 1$, for $e_1 = 0$, $\theta = 0$ are finite. According to (6) when none of the derivatives $\phi'(z_i)$, $(i = 1, 2, 3)$ vanish, J_4 is always finite and clearly

$$\lim_{\theta \doteq 0,\, e_1 \doteq 0,\, e_2 \doteq 0} (J_4) = 0. \tag{12}$$

The denominator of (11) vanishes when any of the derivatives $\phi'(z_i)$, $(i = 1, 2, 3)$ vanish; but in such a case the corresponding derivative may be cancelled from the Jacobian (6), so that on account of (12) the Jacobian (6) always vanishes in case of a combined infinitesimal dilatation and rotation as defined (8) and (9). This is still true in case of either a pure dilatation, or a pure rotation. Conversely, it can be shown without difficulty that it is always possible to determine a dilatation (8) and a rotation (9) in such a manner, that any values of m_1, m_2, m_3, satisfying (6) will be the slopes of the directions of the displacements of A_1, A_2 and A_3.

The condition that three tangents to C at A_1, A_2, A_3 with the slopes m_1, m_2, m_3 are concurrent is

$$a(m_1m_2 - 2m_1m_3 + m_2m_3) + b(m_1 - m_3) = 0. \tag{13}$$

As a and b are positive, the two parentheses in (13) are of opposite sign. Adding the condition $J_4 = 0$, i. e.

$$a(m_1 - m_3) + b(m_1m_2 - 2m_1m_3 + m_2m_3) = 0, \tag{14}$$

(13) and (14) can exist simultaneously only when

$$(m_1m_2 - 2m_1m_3 + m_2m_3)^2 - (m_1 - m_3)^2 = 0. \tag{15}$$

But for $a \neq b$, $|m_1 - m_3| \neq |m_1m_2 - 2m_1m_3 + m_2m_3|$, hence in this case (13) and (14) cannot exist together. When $a = b$, then when (13) is true, (14) is true also. Hence

THEOREM I. *When the Jacobian J_4 (and no others) vanishes for an inscribed rhomb $A_1A_2A_3A_4$, then the tangents to C at A_1, A_2, A_3 are concurrent only when $A_1A_2A_3A_4$ is a square.*

Within the restrictions of this theorem and in all cases where not all four tangents at A_1, A_2, A_3, A_4 are concurrent, the medians M_τ and M_σ associated with the directions of A_1A_3 and A_2A_4 intersect each other singly in the center R of the rhomb (square). In a continuous change of the pair of orthogonal directions (τ, σ) in the neighborhood of A_1A_3 and A_2A_4 the associated inscribed rhombs change continuously. This in connection with the foregoing results leads to

THEOREM II. *If for a certain inscribed rhomb only one of the Jacobians,*

say J_4, vanishes, then the inscribed rhombs in the neighborhood of the given rhomb are continuously connected, and are either infinitesimal dilatations or rotations, or combined infinitesimal dilatations and rotations of the original rhomb, with A_4 stationary.

7. In the system of equations (2) we may also consider z_3, z_2, z_1 successively as the independent variables. Supposing that none of the $\phi'(z_i)$, $i = 1, 2, 3, 4$, vanish at the vertices of the rhomb, which, by properly choosing the coördinate axes, does not imply loss of generality, the vanishing of the corresponding Jacobians is equivalent to

(16) $$J_3 \equiv \begin{vmatrix} 1 & 1 & -1 \\ m_4 & m_1 & -m_2 \\ a & m_1 b & a \end{vmatrix} = 0,$$

or

(17) $$a(2m_1 - m_2 - m_4) + b(m_1 m_2 - m_1 m_4) = 0,$$

(18) $$J_2 \equiv \begin{vmatrix} 1 & 1 & 1 \\ m_3 & m_4 & m_1 \\ -m_3 b & a & m_1 b \end{vmatrix} = 0,$$

or

(19) $$a(m_3 - m_1) + b(m_1 m_4 - 2m_1 m_3 + m_3 m_4) = 0,$$

(20) $$J_1 \equiv \begin{vmatrix} -1 & 1 & 1 \\ -m_2 & m_3 & m_4 \\ a & -m_3 b & a \end{vmatrix} = 0,$$

or

(21) $$a(m_2 + m_4 - 2m_3) + b(m_2 m_3 - m_3 m_4) = 0.$$

The problem now is to find the geometrical meaning of the simultaneous vanishing of two Jacobians, for instance

$$J_4 = 0 \quad \text{and} \quad J_3 = 0.$$

Eliminating a and b between the two equations we find as a necessary condition

(22) $$(m_1 - m_2)[4m_1 m_3 - (m_1 + m_3)(m_2 + m_4)] = 0.$$

The three cases must therefore be considered

(1) $$m_1 = m_2,$$

(2) $$4m_1 m_3 - (m_1 + m_3)(m_2 + m_4) = 0,$$

each of these under the exclusion of the other, and

(3) the existence of (1) and (2) simultaneously.

1. If in $J_4 = 0$ (7) we put $m_2 = m_1$, it reduces to

$$(a + bm_1)(m_1 - m_3) = 0. \tag{23}$$

Similarly $J_3 = 0$ reduces to

$$(a + bm_1)(m_1 - m_4) = 0. \tag{24}$$

As $m_1 = m_2 = m_3 = m_4$ would involve the existence of (2) we exclude this possibility at present. Hence we have as a necessary and sufficient condition for the simultaneous equations $J_4 = 0$, $J_3 = 0$ in case (1)

$$m_1 = m_2 = -\frac{a}{b}. \tag{25}$$

As this condition is otherwise independent of any particular values of m_3 and m_4, it follows that A_3 and A_4 are both stationary and the tangents at A_1 and A_2, Fig. 2, are parallel and are at right angles to the sides A_1A_4

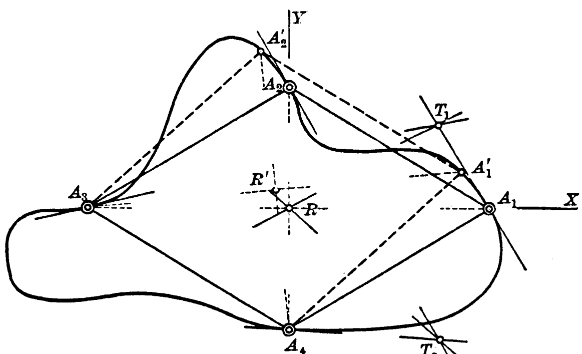

FIG. 2.

and A_2A_3. The tangents T_1R and T_2R to the corresponding medians at R and accordingly these themselves intersect at R. The case where T_1, R, T_2 are collinear will be taken up under (2). We can therefore state

THEOREM III. *The rhombs in the neighborhood of $A_1A_2A_3A_4$ in case* (1) *form a continuous set and the transformation corresponding to this set consists of an infinitesimal rotation of A_1 and A_2 about A_4 and A_3 as instantaneous centers.*

(2) $$4m_1m_3 - (m_1 + m_3)(m_2 + m_4) = 0.$$

In this case the tangents at A_1 and A_3 intersect in a point T_1, those at A_2 and A_4 in a point T_2, such that T_1, R, T_2 are collinear, Fig. 3. To prove this, from the equations

$$y = m_1(x - a),\quad y - b = m_2x,\quad y = m_3(x + a),\quad y + b = m_4x$$

of the tangents at A_1, A_2, A_3, A_4, the coordinates of T_1 and T_2 are found for

$$T_1 \left\{ a\frac{m_1 + m_3}{m_1 - m_3}, \qquad a\frac{2m, m_3}{m_1 - m_3} \right\}$$

and for

$$T_2 \left\{ b\frac{2}{m_4 - m_2}, \qquad b\frac{m_4 + m_2}{m_4 - m_2} \right\}.$$

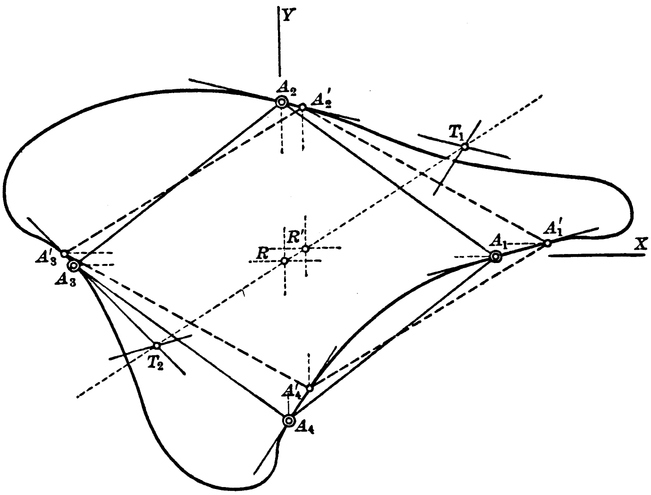

FIG. 3.

The condition for the collinearity of T_1, R, T_2 is

$$\begin{vmatrix} a\dfrac{m_1 + m_3}{m_1 - m_3} & a\dfrac{2m_1m_3}{m_1 - m_3} & 1 \\[2ex] b\dfrac{2}{m_4 - m_2} & b\dfrac{m_2 + m_4}{m_4 - m_2} & 1 \\[2ex] 0 & 0 & 1 \end{vmatrix} = 0,$$

which reduces to

$$4m_1m_3 - (m_1 + m_3)(m_2 + m_4) = 0,$$

which clearly is identical with (2). Conversely it follows easily that when the slopes satisfy this condition, the points, T_1, R, T_2 are concurrent.

As T_1R, T_2R are the tangents to the medians through R, *the latter must be tangent at R.* In the continuous changes of the medians in the neighborhood of R, a tangency arises by the coincidence of two points of intersection. *Geometrically case* (2) *is therefore equivalent to the coincidence of two inscribed rhombs with parallel axes, and there exists continuity of connection of the inscribed rhombs in the neighborhood of the given rhomb.* *The*

infinitesimal transformation consists of a dilatation in the direction of the axes of the rhomb and a translation along the line of collinearity of T_1, R *and* T_2.

It is of course possible that T_1 and T_2 coincide at a point T. If this point is infinitely distant, then $m_1 = m_2 = m_3 = m_4$; the four tangents are parallel. The coexistence of (1) and (2) is equivalent to two parallel tangents at A_1 and A_2 and collinearity of T_1, R and T_2, so that case (3), except when $m_1 = m_2 = m_3 = m_4$, does not yield anything new.

In a similar manner as in case of $J_4 = 0$, $J_3 = 0$, we find for the simultaneous vanishing of any other two of the four Jacobian expressions, $J_i = 0$ and $J_k = 0$, the necessary condition

$$(m_j - m_h)[4m_1m_3 - (m_1 + m_3)(m_2 + m_4)] = 0, \tag{26}$$

where m_j and m_h are the slopes of the tangents at the points A_j and A_h which with A_i and A_k form the rhomb.

If in addition to (26) $J_i = 0$, then also $J_k = 0$. Furthermore, the collinearity of T_1, R, T_2 in connections with $J_i = 0$ also makes the three other Jacobian expressions vanish. Hence the

THEOREM IV: *If the points* T_1, R, T_2 *are collinear, in other words, if the two medians associated with the directions parallel to the axes of a rhomb inscribed to an ordinary closed curve touch each other at the center of the rhomb, then all Jacobians defined in connection with the system of functions* (2) *vanish simultaneously with the system if one of the Jacobians vanishes.*

8. In general, when no particular assumptions about the functions $\phi(t)$ and $\psi(t)$ are made, the system of equations (2) and any of the Jacobians, like (4), vanish simultaneously for a finite number of sets of values z_1, z_2, z_3, z_4 and if no other Jacobian vanishes for any of these sets, then there exists for every corresponding rhomb the condition stated in theorem II. If in addition another Jacobian vanishes, then all other Jacobians vanish, when the points T_1, R, T_2 are collinear, as stated in theorem IV. When only two Jacobians vanish simultaneously with (2), theorem III results. If for an inscribed rhomb the points T_1, R, T_2 are collinear and none of the Jacobians vanish, then the rhombs in the neighborhood of the given rhomb are also continuously connected. This corresponds to the case in which two points of intersection of the medians associated with two orthogonal continuously changing directions coincide.

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